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The Inverse Problem of Stationary Covariance Generation

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The paper considers the problem of passing from a stationary covariance, or spectral matrix, associated with the output of a constant linear finite-dimensional system excited by white noise to the set of all possible systems of minimum possible dimension which will generate this covariance. The problem, originally posed by R. E. Kalman in 1965, is solved by identifying each possible system with the solution of a quadratic matrix inequality; an algorithm for the solution of the inequality is also presented.

KEY WORDS: Linear system; System theory; Minimal realization; State space; Network theory; Spectral density; Quadratic matrix inequalities; Spectral matrix; Transfer function; Covariance generation.

1. INTRODUCTION

A fundamental problem of linear system theory is the following: given an $m \times p$ matrix $W(s)$ of rational functions of a complex variable s, with $W(\infty) < \infty$, determine real constant matrices F, H, L, and J of dimension $n \times n$, $n \times m$, $n \times p$, and $m \times p$ such that

$$W(s) = H'(sI - F)^{-1}L + J \tag{1}$$

(the prime denotes matrix transposition). The above-mentioned problem is motivated by the physically oriented problem of "realizing" a prescribed transfer function matrix $W(s)$ with an analog computer connection, which simulates equations

$$\dot{x} = Fx + Lu \tag{2a}$$

$$y = H'x + Ju \tag{2b}$$

such that $L[y] = W(s)\, L[u]$, the symbol $L[\cdot]$ denoting Laplace transformation.

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The solution to this problem is well known (see, e.g., References 1–4). Also of interest is the problem of determining all sets, rather than one set, of matrices F, H, L, J satisfying (1), i.e., determining all realizations of a prescribed $W(s)$; among all such realizations, those where the dimension of F has the smallest possible value are of special interest. Such realizations are termined minimal, and have special properties. For the study of these matters, see also References 1–4.

Analogous problems arise in the simulation of prescribed spectral density matrices, or equivalently stationary covariance matrices. Thus suppose given a spectral matrix $\Phi(s)$ of rational functions of s, i.e., $\Phi(j\omega)$ is nonnegative definite for all real ω, and $\Phi(s)$ is para-Hermitian, that is, $\Phi(s) = \Phi'(-s)$. One wishes to simulate the spectral matrix Φ, that is, to find a linear system whose output has spectral density Φ when the system input is white noise. Before considering this problem though, it is necessary to understand the associated analysis problem.

If in (2), u becomes white noise, i.e., $E[u(t)\, u'(\tau)] = \delta(t - \tau)$, and if the eigenvalues of F possess negative real parts, then the spectral density of the output y, call it $\Phi(s)$, is given in terms of F, H, L, J by

$$\Phi(s) = W(s)\, W'(-s) \tag{3}$$

where $W(s)$ is related to F, H, L, J by (1). The determination of $\Phi(s)$ from F, H, L, and J is straightforward.

The converse problem (which is the simulation problem) is to start with a prescribed $\Phi(s)$ and arrive at a quadruple F, H, L, J such that equations (1) and (3) hold. The normal procedure would be to find a $W(s)$ satisfying (3), i.e., to perform a spectral factorization using any of the known procedures,[5–7] and then to determine F, H, L, J from $W(s)$ using one of the standard algorithms. Procedures which eliminate some intermediate steps are also available[8–9]; these yield one quadruple of matrices F, H, L, J, which will suffice.

There are however many quadruples F, H, L, J which define an analog computer simulation of $\Phi(s)$, first because there are many $W(s)$ which satisfy (3), and second, because each $W(s)$ can be simulated by an infinity of quadruples F, H, L, J. Let us though impose the natural requirement that simulations only be considered when the F matrix has minimum dimension among the dimensions of all F matrices which will work in some simulation. We shall say that such an F matrix is globally minimal and that F, together with the associated H, L, and J, defines a globally minimal realization of $W(s)$, and, by abuse of language, of $\Phi(s)$. Note that associated with any $W(s)$ satisfying (3) there is a minimal realization of $W(s)$, and, again by abuse of language, this is a realization of $\Phi(s)$. It may not be globally minimal, however, since there may be another $W(s)$ satisfying (3) such that the dimension of the F matrix in a minimal realization of this second $W(s)$ is less than the corresponding quantity for the first $W(s)$.

The multiplicity of minimal dimension quadruples associated with one $W(s)$ is easy to describe: as we know, if F, H, L, J is any one quadruple with minimal dimension F such that (1) holds, all other quadruples with minimal dimension F are given by TFT^{-1}, $(T^{-1})\, H$, TL, J, where T ranges over the set of all nonsingular matrices of the same dimension as F.[2] However, if two quadruples are associated with different transfer functions both satisfying (3), no such simple relation exists among them, even if they are both globally minimal.

The problem which we consider can now be stated: For a prescribed spectral density matrix $\Phi(s)$, describe all quadruples $\{F, H, L, J\}$ with F of globally minimal dimension such that any quadruple defines a simulation of $\Phi(s)$, i.e., such that (1) and (3) hold.

In the next section, we review the frequency-domain (or s-plane) relation existing between various $W(s)$ satisfying (3), and discuss in particular the case where $\Phi(s)$ is a scalar rather than a matrix. Section 3, the longest section, relates using state-space ideas the various $W(s)$ which can satisfy (3); the main conclusion is that all quadruples F, H, L, J which define a $W(s)$ satisfying (3) and which have globally minimal dimension are defined by the solutions of a quadratic matrix inequality. The assumption that one solution of this inequality is known allows its replacement by a much simpler inequality, the solution of which is discussed in Section 4. Limiting solutions of the inequality (corresponding to replacing an inequality sign by an equality sign) define quadruples F, H, L, J for which the L matrix has a minimal number of columns; this is equivalent [see (1)] to the associated $W(s)$ having a minimal number of columns, or [see (2)] to the system generating $\Phi(s)$ having a minimal number of inputs.

Kalman[10] posed a closely related problem to that considered here. He assumed a certain structure for the system generating $\Phi(s)$, a structure derived here as a consequence of global minimality of the F matrix dimension. He restricted $\Phi(s)$ to be such that $\Phi(\infty) = 0$, and sought to find all simulations of $\Phi(s)$.

The assumption that $\Phi(\infty) = 0$ is not a helpful one to make. Indeed in Reference 8, for a related problem, some attention is paid to reformulating a problem where $\Phi(\infty) = 0$ as one where $\Phi(\infty)$ is nonsingular before the problem proper is solved.

2. FREQUENCY DOMAIN RELATIONS AMONG SPECTRAL FACTORS

By way of introduction, consider the problem of finding system transfer function matrices $W(s)$ for which the associated spectral density is

$$\Phi(s) = (2 - s^2)/(1 - s^2) \tag{4}$$

It is well known that matrices $W(s)$ exist which are 1×1, i.e., a single-input system will suffice to generate $\Phi(s)$. Two simple transfer functions are defined by

$$W_1(s) = (\sqrt{2} - s)/(1 + s) \qquad \text{and} \qquad W_2(s) = (\sqrt{2} + s)/(1 + s) \tag{5}$$

These system transfer functions have the same poles, but vary according to the position of their zeros. The zeros however are not arbitrary; as is well known, the magnitude of the zeros is the same, and the argument is determined to within a single ambiguity which fixes the position of the zeros at a point in the half plane $\text{Re}[s] < 0$, or the mirror image point in the half plane $\text{Re}[s] > 0$.

One can write down other 1×1 transfer functions which will work; but it is clear that they can only vary from $W_1(s)$ and $W_2(s)$ by the insertion of common factors in the numerator and denominator, or multiplication by an "all-pass" factor,[11] e.g., $(s-a)(s+a)^{-1}$ or $(s^2 - as + b)(s^2 + as + b)^{-1}$ with $a, b > 0$. There are also transfer function matrices generating $\Phi(s)$ of dimension $1 \times p$, where $p > 1$. Thus the 1×2 matrices

$$W_3(s) = \left[\frac{1-s}{1+s} \quad \frac{1}{1+s}\right] \qquad \text{and} \qquad W_4(s) = \left[\frac{1+\frac{1}{2}s}{1+s} \quad \frac{1-\frac{1}{2}\sqrt{3}\,s}{1+s}\right] \tag{6}$$

will readily be found to satisfy (3), and it is clear that for any $p > 1$, there are an infinity of different $W(s)$ of dimension $1 \times p$ which will satisfy (3).

Aside from the theoretical interest in describing such $W(s)$, study of them is important in network theory, where spectral factors with nonminimal p may be required in the synthesis of a symmetric positive real matrix.[11]

The frequency domain characterization of all the $W(s)$ satisfying (3) has been given by Youla,[5] but see also Reference 12 for further remarks. We summarize the key results in the following lemma.

Lemma 1. Let $\Phi(s)$ be an $m \times m$ spectral density matrix, i.e., $\Phi(j\omega)$ is nonnegative definite for all real ω, $\Phi(s) = \Phi'(-s)$, and each entry of Φ is analytic for $s = j\omega$ for all real ω; suppose every entry of $\Phi(s)$ is a rational function of s, and suppose $\Phi(s)$ has rank r almost everywhere throughout the s-plane. Then, (a) there exists an $m \times r$ matrix $\overline{W}(s)$ of rational functions satisfying (3), with every entry of $\overline{W}(s)$ analytic for $\mathrm{Re}[s] \geqslant 0$, and $\overline{W}(s)$ possessing rank r throughout $\mathrm{Re}[s] > 0$; moreover $\overline{W}(s)$ is unique to within multiplication on the right by an arbitrary real constant orthogonal matrix; (b) any other $m \times p$ matrix $W(s)$ of rational functions satisfying (3) is given by

$$W(s) = \overline{W}(s)\, U(s) \tag{7}$$

where $U(s)$ is an $r \times p$ matrix of rational functions satisfying

$$U(s)\, U'(-s) = I \tag{8}$$

Conversely if $U(s)$ is a matrix satisfying (8), then $W(s)$ as defined by (7) satisfies equation (3). ■

Note that when $\Phi(s)$ is a scalar, r becomes unity, and the matrix $\overline{W}(s)$ becomes a scalar function with the "minimum phase" property.[11] Other 1×1 matrices $W(s)$ satisfying (3) then follow by multiplying $\overline{W}(s)$ by an "all-pass" function.[11] If the poles of this function do not cancel zeros of $\overline{W}(s)$, then $W(s)$ has more poles than $\overline{W}(s)$. Then the dimension of the matrix F_w in any minimal realization F_w, H_w, L_w, J_w of $W(s)$ will exceed the dimension of the matrix $F_{\overline{w}}$ in any minimal realization $F_{\overline{w}}$, $H_{\overline{w}}$, $L_{\overline{w}}$, $J_{\overline{w}}$ of $\overline{W}(s)$. (Thus, relative to the dimensions of all possible F matrices in all possible quadruples realizing $\Phi(s)$, F_w is not globally minimal.) On the other hand, if the poles of the all-pass function $U(s)$ do cancel zeros of $\overline{W}(s)$, then $W(s)$ becomes essentially $\overline{W}(s)$ with some zeros switched from the half plane $\mathrm{Re}[s] < 0$ to the half plane $\mathrm{Re}[s] > 0$. Note also that for the purpose of defining a system which will generate $\Phi(s)$ [as distinct from finding all $W(s)$ satisfying (3)], one requires $W(s)$

to have elements analytic in the half plane $\text{Re}[s] \geqslant 0$. This requires $U(s)$ in (7) not to introduce into $W(s)$ elements with a pole in $\text{Re}[s] \geqslant 0$.

One can thus state in rough terms how to get all 1×1 $W(s)$ which have an associated globally minimal realization. Start with the $\overline{W}(s)$ of Lemma 1; then transferral of any one zero from $\text{Re}[s] < 0$ to the mirror image point in $\text{Re}[s] > 0$ (together with its complex conjugate if it is not a real zero) generates a $W(s)$ of the desired type; intuitively one sees that all 1×1 $W(s)$ are generated this way. Thus there are a finite number of such $W(s)$, in contrast to the fact that there are an infinity of $W(s)$ possessing globally minimal realizations which are $1 \times p$, $p > 1$. (The actual number of 1×1 $W(s)$ will be at most 2^n, if $\overline{W}(s)$ has n zeros; it will be less if some of these zeros are complex.)

For the purposes of this paper, we shall impose for most of the results further constraints on Φ beyond those indicated in the lemma. We shall require that $\Phi(s)$ have full rank almost everywhere, and that $\Phi(\infty)$ be nonsingular. The theory is extendable to deal with the situation where the extra constraints fail but the procedure is somewhat intricate, not especially illuminating, and has in any case been covered before.[8]

With the extra constraints, conclusion (*a*) of the lemma can be strengthened to deduce that $\overline{W}(s)$ is nonsingular throughout the half plane $\text{Re}[s] > 0$.

Associated with any spectral density matrix $\Phi(s)$ there exists a positive real matrix $Z(s)$, derivable as follows.[11] Each element $\phi_{ij}(s)$ of $\phi(s)$ may be expanded as a sum of partial fractions and a term $\phi_{ij}(\infty)$. Those partial fractions with poles in $\text{Re}[s] < 0$ may then be summed together, and when added to $\frac{1}{2}\phi_{ij}(\infty)$, yield the i–j entry of $Z(s)$. The para-Hermitian property of $\Phi(s)$ then guarantees that the sum of the partial fractions with poles in $\text{Re}[s] > 0$ and $\frac{1}{2}\phi_{ij}(\infty)$ yields the j–i entry of $Z(-s)$. Thus

$$\Phi(s) = Z(s) + Z'(-s) \tag{9}$$

That $Z(s)$ is positive real follows from the constraint on the poles of its elements, and the nonnegativity of $\Phi(j\omega)$ for all real ω, see Reference 11. For future reference, we note that $Z(\infty) = Z'(\infty) = \frac{1}{2}\Phi(\infty)$.

Thus the earlier stated problem is one of finding all globally minimal dimension quadruples F, H, L, J such that the matrix $W(s)$ defined by (1) satisfies

$$Z(s) + Z'(-s) = W(s)\, W'(-s) \tag{10}$$

Of course, in view of the well-known connections between minimal realizations of the one transfer function matrix, it suffices for each $W(s)$ possessing a globally minimal realization to indicate only one such realization.

3. MAIN RESULTS

The results of this section may be summed up as follows. We shall show first that all $W(s)$ satisfying (10) which possess globally minimal realizations have realizations with the same F and H matrices (but in general different L and J matrices). Then we shall consider how these L and J matrices may be generated, using a minimal realization of $Z(s)$ in (10), together with an additional matrix essentially playing the role of a parameter. Some equations will be derived which implicitly define all possible L and J, but are not in a form which permits the explicit construction of all possible L and J. These equations are known from the Kalman–Yakubovich lemma.[13,14]

The equations of the Kalman–Yakubovich lemma are then restructured so that the determination of all L and J matrices required the solution of a quadratic matrix inequality.

Lemma 2. Let $Z(s)$ be a positive real matrix derived from a spectral density matrix $\Phi(s)$ by the procedure outlined at the end of the last section, and let $Z(s)$ have a minimal realization F, H, G, J. Let $W(s)$ satisfy (10) and possess a globally minimal realization. Then $W(s)$ has a realization of the form F, H, L_w, J_w; moreover, there exists a real positive definite matrix P_w such that the following equations hold:

$$P_wF' + FP_w = -L_wL_w' \tag{11a}$$

$$P_wH = G - L_wJ_w' \tag{11b}$$

$$2J = J_wJ_w' \tag{11c}$$

Conversely if real matrices P_w, L_w, and J_w can be found such that (11) hold with P_w positive definite symmetric, then $W(s) = H'(sI - F)^{-1}L_w + J_w$ satisfies (1) and the dimension of F is globally minimal. ■

Proof. Let $\delta[N(s)]$ denote the degree[1] of a transfer function matrix $N(s)$. If $N(\infty) < \infty$, this is the dimension of the F matrix in any minimal realization of $N(s)$. Then because the poles of elements of $Z(s)$ and $Z'(-s)$ cannot coincide, being restricted to half planes $\text{Re}[s] < 0$ and $\text{Re}[s] > 0$ respectively, $\delta[\Phi(s)] = 2\delta[Z(s)]$ by a well-known property of degree. A further well-known property of degree is that

$$\delta[W(s)\, W'(-s)] \leqslant 2\delta[W(s)],$$

and thus for any $W(s)$ satisfying (10), it must be true that $\delta[W(s)] \geqslant \delta[Z(s)]$.

Reference 14 demonstrates the existence of one $W(s)$ satisfying (10) with the property that $\delta[W(s)] = \delta[Z(s)]$, and thus we conclude that any $W(s)$ satisfying (10) which possesses a globally minimal realization must have $\delta[W(s)] = \delta[Z(s)]$. Trivial modifications of Lemmas 7 and 8 of Reference 14 yield that any $W(s)$ for which $\delta[W(s)] = \delta[Z(s)]$ and (10) holds has a minimal realization of the form F, H, L_w, J_w. This proves the first part of the lemma.

Define now the matrix P_w by

$$P_wF' + FP_w = -L_wL_w' \tag{11a}$$

Since $W(s)$ has elements with poles in $\text{Re}[s] < 0$, and F is of minimum dimension, we know that the eigenvalues of F have negative real part. Since moreover the pair F, L_w is completely controllable, a consequence of the minimality of F, Reference 13 allows us to conclude that P_w is (well-) defined uniquely from (11a) and is positive definite symmetric. It remains to verify (11b) and (11c). Now

$$W(s)\, W'(-s) = J_w J_w' + H'(sI - F)^{-1} L_w J_w' + J_w L_w'(-sI - F')^{-1} H$$
$$+ H'(sI - F)^{-1} L_w L_w'(-sI - F')^{-1} H$$

The last term may be rewritten using (11a) to yield

$$H'(sI - F)^{-1} L_w L_w'(-sI - F')^{-1} H = H'(sI - F)^{-1} P_w H + H' P_w(-sI - F')^{-1} H$$

so that then

$$W(s) W'(-s) = J_w J_w' + H'(sI - F)^{-1}(L_w J_w' + P_w H) + (L_w J_w' + P_w H)'(-sI - F')^{-1} H$$

But (10) provides another expression for $W(s)\, W'(-s)$ in terms of the matrix quadruple realizing $Z(s)$, viz., F, H, G, J. Thus

$$W(s)\, W'(-s) = 2J + H'(sI - F)^{-1} G + G'(-sI - F')^{-1} H$$

Setting $s = \infty$, one recovers (11c). A decomposition of $W(s)\, W'(-s)$ like that used to obtain $Z(s)$ from $\Phi(s)$ yeilds that $H'(sI - F)^{-1} G = H'(sI - F)^{-1}(L_w J_w' + P_w H)$. Then we note that, because F, H is a completely observable pair, $G = L_w J_w' + P_w H$, which is (11b). This proves the second part of the lemma.

Finally, suppose $W(s) = H'(sI - F)^{-1} L_w + J_w$, and equations (11) hold for some positive definite P_w. The second expression above for $W(s)\, W'(-s)$ follows from the definition of $W(s)$ and from (11a), and the third follows on using (11b) and (11c). But this expression is precisely (10), with $Z(s)$ expanded using a quadruple realizing it. Global minimality is obvious. Thus the final part of the lemma is proved.

Lemma 2 reduces to an algebraic problem the determination of all $W(s)$ possessing a globally minimal realization and satisfying (3). The algebraic problem is to "solve" the three equations in (11) for the unknown matrices P_w, L_w, and J_w given the known matrices F, H, G, and J.

As the next step in the argument, we shall now indicate a quadratic matrix inequality equivalent to (11); this is not the inequality whose solution is studied in the next section, but is more complicated and presumably harder to solve. By way of notation, $A \geqslant 0$ (>0) for a symmetric matrix A will mean A is nonnegative (positive) definite, and $A \geqslant B$ will mean that $A - B \geqslant 0$.

We now include the assumption that $\Phi(\infty)$ is nonsingular, and have:

Theorem 1. With the same hypothesis as Lemma 2 and the assumption that J is nonsingular, the solution of (11) is equivalent to the solution of

$$P_w F' + F P_w \leqslant -(P_w H - G)(2J)^{-1}(P_w H - G)' \tag{12}$$

in the sense that if (11) hold, then P_w satisfies (12), and if (12) holds for some symmetric P_w, then P_w is positive definite, and associated L_w and J_w exist which with P_w satisfy (11). ∎

Proof. Observe first that if $J_w J_w'$ is nonsingular, $M = J_w'(J_w J_w')^{-1} J_w \leqslant I$. For if x is an eigenvector of M, $\lambda x = Mx$ for some λ. Then $\lambda(J_w x) = J_w(\lambda x) = J_w(Mx) = (J_w M)\, x = J_w x$ since $J_w M = J_w$ by inspection. Hence $\lambda = 1$, or $J_w x = 0$ and thus $\lambda = 0$. Thus the eigenvalues of $J_w'(J_w J_w')^{-1} J_w$ are all either 0 or 1, which establishes the inequality. Note that if J_w is nonsingular, equality holds.

Now suppose equations (11) hold. Then, as required,

$$P_w F' + F P_w \leqslant -L_w J_w'(J_w J_w')^{-1} J_w L_w' = -(P_w H - G)(2J)^{-1}(P_w H - G)'$$

Observe that equality holds if J_w is nonsingular, which will be the case if J_w is square' i.e., the associated $W(s)$ has minimum number of columns.

Now suppose (12) holds for some P_w. Then the matrix

$$P_w F' + F P_w + (P_w H - G)(2J)^{-1}(P_w H - G)'$$

is nonpositive definite, so that there exists a real constant matrix N such that

$$P_w F' + F P_w + (P_w H - G)(2J)^{-1}(P_w H - G)' = -NN'$$

(The matrix N is determined only to within multiplication on the right by an arbitary real constant matrix $\tilde{V}$ satisfying $\tilde{V}\tilde{V}' = I$.) Then in order that equations (11) hold, it is necessary and sufficient that

$$L_w = [-(P_w H - G)(2J)^{-1/2} \qquad M]\, V$$

and

$$J_w = [(2J)^{1/2} \qquad 0]\, V$$

where V is any real constant matrix for which $VV' = I$, and the block of zeros in J_w is of such a size that the number of columns of ϕ_w and of J_w are equal.

To see that P_w is positive definite, suppose this is not the case. Then the pair F, L_w could not be completely observable. The manipulations of Lemma 2 would still guarantee that $W(s) = J_w + L_w'(sI - F)^{-1} G$ satisfied $Z(s) + Z'(-s) = W(s)\, W'(-s)$, but now, since F, L_w is not completely observable, $\delta[W(s)]$ is less than the dimension of F, i.e., $\delta[Z(s)]$. The proof of Lemma 2 shows that this is impossible. Thus by contradiction, P_w is positive definite.

Theorem 1 reduces the problem of determining all solutions of (11) to the problem of solving (12). Although each solution P_w of (12) determines an infinity of L_w and J_w, it is important to note that these vary but trivially from one another. Thus if P_w determines, say, L_{w1}, J_{w1} yielding $W_1(s) = J_{w1} + H'(sI - F)^{-1} L_{w1}$, then all other $W(s)$ determined by the same P_w are given by $W_1(s)\, V$, where V is a real constant matrix for which $VV' = I$.

Rather than pursuing the problem of directly solving (12), we shall adopt a modified line of attack. This will require us to assume knowledge of one solution of (12), call it $\bar{P}$, and then to seek the solution P_w as defined by $Q = P_w - \bar{P}$. We have the following result:

Theorem 2. Let a symmetric $\bar{P}$ satisfy (12) with equality, and let P_w be any other symmetric matrix satisfying the inequality (12). Define the matrix Q by $Q = P_w - \bar{P}$. Then Q satisfies

$$Q\bar{F}' + \bar{F}Q \leqslant -QH(2J)^{-1}H'Q \tag{13}$$

where

$$\bar{F} = F + (\bar{P}H - G)(2J)^{-1}H' \tag{14}$$

Conversely, if $\bar{P}$ satisfies (12) with equality, and Q satisfies (13), then $P_w = Q + \bar{P}$ satisfies (12). Finally, equality in (13) holds if and only if the corresponding P_w satisfies (12) with equality. ■

Proof. The proof is straightforward, relying on simple manipulations of (12).

In the next section, we shall discuss procedures for solving (13). Meanwhile, we shall present several additional results stemming from (13).

Corollary 1. With the same hypothesis as Theorem 1, there exists a matrix $\bar{P}$ satisfying (11) and (12) such that for any other solution P_w, $P_w - \bar{P} \geqslant 0$. This matrix $\bar{P}$ defines the family of spectral factors $\bar{W}(s)$ of Lemma 1 for which $\bar{W}(s)$ is nonsingular throughout $\text{Re}[s] > 0$. (Recall that two members of the family differ only by multiplication on the right by an arbitrary real constant orthogonal matrix.) ■

Proof. Define $\bar{P}$ as the matrix solution of (12) which generates a $\bar{W}(s)$. Then $\bar{P}$ satisfies (12) with equality, since $\bar{W}(s)$ has a minimum number of columns. Now $\bar{W}(s)$, being given by, say, $\bar{J} + H'(sI - F)^{-1}\bar{L}$ has inverse $\bar{W}^{-1}$, which is readily computed as

$$\bar{W}^{-1}(s) = \bar{J}^{-1}H'(sI - F + \bar{L}\bar{J}^{-1}H')^{-1}\bar{L}\bar{J}^{-1} \tag{15}$$

Now a well-known property of degree guarantees that $\delta[\bar{W}^{-1}(s)] = \delta[\bar{W}(s)]$, and so the dimension of $F - \bar{L}\bar{J}^{-1}H'$ in the realization of $\bar{W}^{-1}(s)$ defined by (15) is minimal. Because $\bar{W}(s)$ is nonsingular throughout $\text{Re}[s] > 0$, it follows that the eigenvalues of $F - \bar{L}\bar{J}^{-1}H'$ have nonpositive real parts. Recalling the definitions of $\bar{L}$ and $\bar{J}$ in terms of $\bar{P}$, as given in the course of the proof of Theorem 1, we note that an equivalent statement is that the eigenvalues of $\bar{F} = F + (\bar{P}H - G)(2J)^{-1}H'$ have nonpositive real parts. Now (13) allows us to relate $\bar{P}$ to P_w; since any solution of (13) defines a matrix K such that

$$Q\bar{F}' + \bar{F}Q = -KK' \tag{16}$$

the eigenvalue restriction on $\bar{F}$ guarantees that Q is nonnegative definite, via a variant of the lemma of Lyapunov.[13] In other words, $P_w - \bar{P} \geqslant 0$ as required.

The matrix K in (16) has additional significance. We recall from Lemma 1 that any spectral factor $W(s)$ can be related to $\bar{W}(s)$ via $W(s) = \bar{W}(s)\, U(s)$, where $U(s)$ satisfies $U(s)\, U'(-s) = I$. Suppose that $W(s)$ is defined via P_w, L_w, and J_w, while $\bar{W}(s)$ is as above, in Corollary 1. Then the matrix $U(s)$ can be checked to be

$$U(s) = \bar{J}^{-1}[[\bar{J} : 0]\, V - H'(sI - \bar{F})^{-1}K] \tag{17}$$

where V is a real constant matrix satisfying $VV' = I$.

One might well ask how $\bar{P}$ can be determined. Actually, several ways are available. Thus one can find $\bar{W}(s)$ by standard procedures,[5–7] and from it find $\bar{P}$ (see Reference 14); again, one can give an algorithm for solving (12) with the inequality replaced by equality, which at the same time yields $\bar{P}$ rather than any other solution of the equality (see Reference 8); finally, one can solve a matrix Riccati differential equation, a limiting solution of which satisfies (12) (see Reference 15).

4. SOLUTION OF THE MATRIX INEQUALITY

This section is devoted to the study of inequality (13), restated for convenience as

$$Q\bar{F}' + \bar{F}Q \leqslant -QH(2J)^{-1}H'Q \tag{13}$$

As we know, each solution of this inequality defines a family of globally minimal realizations of a prescribed $\Phi(s)$. We recall that $\bar{F}$ involves the matrix $\bar{P}$, which is required to be one solution of the quadratic matrix inequality (12) that is equivalent to the Kalman–Yakubovich equations; for convenience we shall assume throughout this section that $\bar{P}$ is as defined in Corollary 1, i.e., $\bar{P}$ enables us to define a globally minimal realization of $\Phi(s)$ and of $\bar{W}(s)$, as defined in Lemma 1. This means that with this definition of $\bar{P}$, all solutions of (13) are nonnegative definite, and the matrix $\bar{F}$ has eigenvalues with nonpositive real parts, as explained in the proof of Corollary 1.

We shall consider first the solution of (13) for nonsingular Q, and then for singular Q; finally, we shall note some properties of solutions of (13).

4.1. Nonsingular Solutions of the Matrix Inequality

All nonsingular Solutions of (13) are generated from all nonsingular solutions of

$$Q\bar{F}' + \bar{F}Q = -QH(2J)^{-1}H'Q - R \tag{18}$$

where R ranges over the set of all nonnegative definite matrices; precisely on account of the nonsingularity of Q, the set of nonsingular solutions to (18) is equivalent to the set of nonsingular solutions of

$$Q\bar{F}' + \bar{F}Q = -Q[H(2J)^{-1}H' + S]\,Q \tag{19}$$

or again, to the set of nonsingular solutions of

$$\bar{F}'Q^{-1} + Q^{-1}\bar{F} = -H(2J)^{-1}H' - S \tag{20}$$

where S ranges over the set of all nonnegative definite matrices. Solution procedures for (20) are well known (see, e.g., Reference 16).

It is helpful to note a simple condition which governs whether or not (20) defines invertible matrices Q^{-1} (for one might have all solutions X of

$$\bar{F}'X + X\bar{F} = -H(2J)^{-1}H' - S$$

singular, and these would not yield matrices Q). This condition is that $\bar{F}$ have eigenvalues with negative real parts, as distinct from merely nonpositive real parts. (Equivalently, $\overline{W}(s)$ is required to be nonsingular in $\mathrm{Re}[s] \geqslant 0$, rather than $\mathrm{Re}[s] > 0$, and $\Phi(j\omega)$ is required to be positive, rather than nonnegative, definite, for all real ω.) To see this, note that the nonsingularity of Q^{-1} and its positive definite nature guarantee that $x'Q^{-1}x$ is positive definite for all x along a trajectory of $\dot{x} = \bar{F}x$. The derivative of $x'Q^{-1}x$ along a trajectory is nonpositive, being less than $-x'H(2J)^{-1}H'x$. But now because the pair F, H is completely observable, the pair $\bar{F}$, H is completely observable [see (14)]. Then the derivative of $x'Q^{-1}x$ will not be identically zero along a trajectory, and $\dot{x} = \bar{F}x$ must then be asymptotically stable.[17]

4.2. Singular Solutions of the Matrix Inequality

We now turn to the determination of singular solutions of (13). We exhibit first a simple property of such solutions, and observe how all of them may be generated. However, the procedure for generating all solutions is not systematic (in that one solution can be generated from a number of starting points). Accordingly, we then present a more systematic means of generating all singular solutions to (13).

We start by assuming knowledge of a solution Q to (13) which is singular. There exists then a nonsingular matrix T such that $Q = T\left[\begin{smallmatrix} Q_1 & 0 \\ 0 & 0 \end{smallmatrix}\right] T'$, where Q_1 is nonsingular and symmetric. Then (13) yields

$$\begin{aligned} &\begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} T'\bar{F}'T'^{-1} + T^{-1}\bar{F}T \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} \\ &\qquad = - \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} T'H(2J)^{-1} H'T \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} - T^{-1}RT'^{-1} \end{aligned} \tag{21}$$

where R is a nonnegative definite matrix. Define now

$$T^{-1}\bar{F}T = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \tag{22}$$

It follows readily from (21) that

$$\begin{aligned} &\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F'_{11} & F'_{21} \\ F'_{12} & F'_{22} \end{bmatrix} \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ &\qquad = - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T'H(2J)^{-1} H'T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & I \end{bmatrix} T^{-1}RT'^{-1} \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & I \end{bmatrix} \\ &\qquad = - \begin{bmatrix} H_1 H_1' & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} R_{11} & R_{12} \\ R'_{12} & R_{22} \end{bmatrix} \end{aligned} \tag{23}$$

with appropriate definitions of H_1, R_{11}, etc. On evaluating the products on the left side of (23), and equating the top left-hand corners of the matrix sums on each side of (23), one obtains

$$F_{11}Q_1^{-1} + Q_1^{-1}F_{11} = -H_1H_1' - R_{11} \tag{24}$$

where R_{11} will be nonnegative definite. Equating the bottom right-hand corners yields $R_{22} = 0$, which implies because of the nonnegative nature of R that $R_{12} = 0$. Finally, (23) yields

$$F_{21} = 0 \tag{25}$$

A summary of the above analysis is as follows. A singular solution to (13) is transformed via a congruence transformation to display a maximum amount of main diagonal zeros. The associated transformation of $\bar{F}$ introduces a corresponding block of zeros into the transformed $\bar{F}$. Finally, upon extracting a diagonal block from the quasi-upper-triangular[16] transformed $\bar{F}$, (24) is solved for any nonnegative definite R_{11}. Note that the eigenvalues of F_{11} are a subset of those of $\bar{F}$, and thus (24) may not always yield a nonsingular solution (i.e., if $R_{11} = 0$ and F_{11} has eigenvalues with zero real part, $F_{11}'X + XF_{11} = -H_1H_1'$ may only have a singular solution X, or even no solution).

The procedure outlined via (21)–(25) is reversible, i.e., by tracing it backward one can find all Q satisfying (13) or, equivalently, (26). One finds all similarity transformations T on $\bar{F}$ reducing it to quasi-upper-triangular form [see (22) and (25)], and then finds for all nonnegative definite R_{11} matrices Q_1^{-1} using (24). Then $T\left[\begin{smallmatrix} Q_1 & 0 \\ 0 & 0 \end{smallmatrix}\right] T'$ is a solution of (13).

The problem with generating all solutions of (13) in this way is that numbers of different versions of (24) may field the same Q. To see this, simply note that there exist many matrices T for which Q may be written as $T\left[\begin{smallmatrix} Q_1 & 0 \\ 0 & 0 \end{smallmatrix}\right] T'$ with T, Q_1 nonsingular. Each different T yields a different version of (24); thus starting with any of these versions and working backward leads to the same Q.

One natural way out of this difficulty is to work with a transformed version of (13). For simplicity, assume all eigenvalues of $\bar{F}$ are real and distinct. (The theory is extendable from this case.) Then there exists a nonsingular matrix T such that $\hat{F} = T^{-1}\bar{F}T$ is diagonal. Then with $\hat{H} = T'H(2J)^{-1/2}$, each solution $\hat{Q}$ of

$$\hat{Q}\hat{F} + \hat{F}\hat{Q} = -\hat{Q}\hat{H}\hat{H}'\hat{Q} - \hat{R} \tag{26}$$

(where $\hat{R}$ ranges over the set of nonnegative definite symmetric matrices) defines a solution of (13) given by $Q = T\hat{Q}T'$. (The matrix $\hat{R}$ in (26) is related to R in (18) by $R = T\hat{R}T'$). Conversely, each solution of (13) defines R and a solution of (26).

The simple form of $\hat{F}$ in (26) makes the determination of all singular solutions comparatively simple. As a preliminary we note the following:

Lemma 3. Suppose in (26) that $\hat{Q}$ is singular, and let $x \neq 0$ be such that $\hat{Q}x = 0$. Then $\hat{R}x = 0$, and $\hat{Q}\hat{F}^i x = 0$ for all positive integers i. ∎

Proof. Multiply (26) on the left by x' and on the right by x. Then $x'\hat{R}x = 0$, which implies $\hat{R}x = 0$ since $\hat{R}$ is nonnegative definite symmetric. Then multiply (26) on the right by x to obtain $\hat{Q}\hat{F}x = 0$. Now apply the same argument to $\hat{F}x$ as was applied to x to conclude in turn that $\hat{Q}\hat{F}^2x = 0$, $\hat{Q}\hat{F}^3x = 0$, etc. This proves the lemma.

The fact that $\hat{Q}x = 0$ implies $\hat{Q}\hat{F}^i x = 0$ for all positive integers i implies that the vector x must have zero entries if $\hat{Q}$ is not zero. For if x has no zero entry, the set of vectors $x, \hat{F}x, \hat{F}^2x, \ldots$ spans the whole space of vectors of dimension equal to the size of $\hat{Q}$.[2] Thus every vector is in the null space of $\hat{Q}$, which implies that $\hat{Q}$ is zero.

Assume then that precisely r entries of x are zero for some x such that $\hat{Q}x = 0$. By recordering the rows and columns of $\hat{Q}$, $\hat{F}$, $\hat{H}$, and $\hat{S}$, we may suppose these entries are the first r. With x of dimension n, this means that the last $(n - r)$ entries of x are nonzero, and then it may be checked that the set $x, \hat{F}x, \hat{F}^2x, \ldots$ spans the subspace of vectors whose first r entries are zero, and whose last $n - r$ entries are arbitrary. Since any vector in this subspace is in the null space of $\hat{Q}$ and of $\hat{R}$, it follows that the last $n - r$ columns of $\hat{Q}$ and $\hat{R}$ are zero. The symmetry of $\hat{Q}$ and $\hat{R}$ then implies that the last $n - r$ rows of each matrix are also zero. Thus (26) becomes

$$\hat{Q}_1\hat{F}_1 + \hat{F}_1\hat{Q}_1 = -\hat{Q}_1(\hat{H}\hat{H}')_1\,\hat{Q}_1 - \hat{R}_1 \tag{27}$$

where the subscript 1 on a matrix denotes the removal of the last $(n - r)$ rows and columns of the matrix. The solution $\hat{Q}$ of (26) (satisfying $\hat{Q}x = 0$) is given by

$$\hat{Q} = \begin{bmatrix} \hat{Q}_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{28}$$

All nonsingular solutions of (27) now follow easily from

$$\hat{F}_1\hat{Q}_1^{-1} + \hat{Q}_1^{-1}\hat{F}_1 = -(\hat{H}\hat{H}')1 - \hat{S}_1 \tag{29}$$

where $\hat{S}_1$ ranges through the set of $r \times r$ nonnegative definite matrices.

All solutions of (26) may be derived in this fashion, that is, by dropping rows and columns from (26) and solving the resulting equation, assuming a nonsingular solution. The point is that whereas reduction of Q via a congruency transformation was necessary in solving (13) as a preliminary step, this is no longer necessary in solving (26).

4.3. Some Additional Points

There is at most one nonsingular Q satisfying (13) with equality. It is given by solving (18) with R set equal to zero, and it generates an interesting transfer function family (members of the family differing by multiplication on the right by an arbitrary real constant orthogonal matrix). Denote a typical member of the family by $\overline{W}(s)$; then by proceeding along lines similar to those used in the proof of Corollary 1, $\overline{W}^{-1}(s)$ is readily found to have the poles of its elements located at the eigenvalues of

$$\bar{\bar{F}} = F + [(\bar{P} + Q)\,H - G](2J)^{-1}H' = \bar{F} + QH(2J)^{-1}H'$$

Hence it follows that

$$Q\bar{\bar{F}}' + \bar{\bar{F}}Q = QH(2J)^{-1}H'Q$$

or

$$Q^{-1}\bar{\bar{F}} + \bar{\bar{F}}'Q^{-1} = H(2J)^{-1}H'$$

The positive definiteness of Q and the easily proved complete observability of the pair $\bar{F}$, H then show that $\bar{F}$ has all its eigenvalues in $\mathrm{Re}[s] > 0$, i.e., that $\overline{W}(s)$ is nonsingular throughout $\mathrm{Re}[s] \leqslant 0$. Such $\overline{W}(s)$ are equally as important as the $\overline{W}(s)$ of Lemma 1.

In the determination of singular solutions Q of (13) via the method based on (26), the situation is rather more complicated when $\hat{F}$ has repeated or complex eigenvalues. Then the structure of singular solutions of (26), and in particular the location of the block of zeros in them, is not so simple. It should also be noted that the solution of (29) implicitly relies on $\hat{F}_1$ possessing all eigenvalues of negative real part, for the same reasons requiring $\bar{F}$ in (18) to have all eigenvalues of negative real part to guarantee nonsingularity of Q.

A further point of interest is that solutions of (13) fall naturally into families, with each family characterized by a solution of (13) with the equality sign. This occurs in the following way. Let Q be an arbitrary solution of (13), and let $\hat{Q}$ be $T^{-1}QT'^{-1}$, so that $\hat{Q}$ satisfies (26) for some $\hat{R}$. Then a number of rows and columns of $\hat{Q}$ will be zero, and with an appropriate reordering of the rows and columns, a nonsingular submatrix $\hat{Q}_1$ of $\hat{Q}$ satisfies (27). Now define $\underline{\hat{Q}}_1$ as the nonsingular solution of

$$\underline{\hat{Q}}_1\hat{F}_1 + \hat{F}_1\underline{\hat{Q}}_1 = -\underline{\hat{Q}}_1(\hat{H}\hat{H}')_1\underline{\hat{Q}}_1 \tag{30}$$

It is not hard to demonstrate that $\underline{\hat{Q}}_1$ exists and that $\underline{\hat{Q}}_1 \geqslant \hat{Q}$. Now by reversing the procedure used for obtaining $\underline{\hat{Q}}_1$ from $\hat{Q}$, we construct a matrix $\underline{\hat{Q}}$ which satisfies (26) with $\hat{R} = 0$. Of course $\underline{\hat{Q}} \geqslant \hat{Q}$. Finally $\underline{Q} = T\underline{\hat{Q}}T'$ yields a matrix satisfying (13) with equality, and such that $\underline{Q} \geqslant Q$.

Reid[16] has shown for equations (13) with the equality sign that if $\underline{\underline{Q}}$ is a nonsingular solution and $\underline{Q}$ any other solution, then $\underline{\underline{Q}} \geqslant \underline{Q}$. This result together with Corollary 1 allows us then to conclude that if there is a nonsingular solution $\underline{\underline{Q}}$ of (13) with the equality sign, then $\underline{\underline{Q}} \geqslant Q \geqslant 0$ for any solution Q of the inequality.

5. CONCLUSIONS

The determination of globally minimal realizations of a prescribed spectral density matrix $\Phi(s)$ has been shown to be equivalent to the solution of a quadratic matrix inequality which is apparently difficult to solve. However, by assuming knowledge of one solution of this inequality, the determination of all solutions of the inequality can be achieved by solving a simpler quadratic matrix inequality.

Solution procedures for this latter inequality have been discussed. It has been shown that the solutions fall into families characterized by limiting solutions of the inequality, obtained by solving it with an equality sign replacing the inequality sign. Upper and lower bounds on the solution to the inequality have also been derived.

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